

[Exact Differential Equations]

22

definition: — A differential equation is said to be exact if it can be derived from its primitive (solution) by directly differentiation.

e.g. 1. $x dy + y dx = 0$ and
its solution be $xy = C$

2. $\sin x \cdot \cos y dy + \cos x \cdot \sin y dx = 0$ and
its solution be $\sin x \cdot \cos y = C$

Thus, we see that the diff. eqn. can be obtained by directly differentiating its solution.

• **Theorem:** — The necessary and sufficient condition that the differential equation

$$M dx + N dy = 0$$

where M and N are functions of x and y
is exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof: — First of all, we show that if
• **Necessary condition:** — $M dx + N dy = 0$... then ... (1)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Let the solution of (1) be $u(x, y) = C$. — (2)

If (1) is exact then it can be obtained by
directly its solution

Differentiating partially (2), we get

$$\frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \dots \dots \dots \quad (3)$$

31

Comparing (1) and (3), we get

$$M = \frac{\partial u}{\partial x} \text{ and } N = \frac{\partial u}{\partial y} \text{ so that}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial x} \text{ and}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

$$\therefore \frac{\partial^2 u}{\partial y \cdot \partial x} = \frac{\partial^2 u}{\partial x \cdot \partial y}$$

$$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

That, the condition is necessary has been proved.

• (ii) Sufficient condition: — Given that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ then we shall prove that}$$

$$M dx + N dy = 0$$

$$\text{Let } \int N dx = \phi \quad \therefore M = \frac{\partial \phi}{\partial x}$$

$$\text{Now, } \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right)$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) \quad \therefore N = \frac{\partial \phi}{\partial y} + \psi(x)$$

where ψ is a fn. of x .

$$\begin{aligned} \therefore M dx + N dy &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \psi(x) dy \\ &= d\phi + \psi(x) dy = d\{\phi + f(x)\} \end{aligned}$$

This shows that $M dx + N dy = 0$ is an exact eqn.

Q. (1) Solve:-

$$(y+x)dx + xdy = 0$$

Solution:- The given diff. eqn. can be written as

$$ydx + xdx + xdy = 0$$

$$\Rightarrow (xdy + ydx) + xdx = 0 \dots \dots \dots (1)$$

Here, $M = y+x$ and $N = x$

$$\text{then } \frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

\therefore The given diff. eqn. be an exact.

$$\text{Now, (1)} \Rightarrow d(xy) + xdx = 0$$

$$\text{Integrating, } xy + \frac{x^2}{2} = K$$

$$\therefore \boxed{x^2 + 2xy = C}$$

which is the required solution.

$$Q. (2) (ax+by+g)dx + (bx+ay+f)dy = 0$$

Solution:- Here $M = ax+by+g$ and
 $N = bx+ay+f$

$$\text{Now, } \frac{\partial M}{\partial y} = b \text{ and } \frac{\partial N}{\partial x} = a$$

[\because This is partial derivative and in this case when we differentiate w.r.t. x then other variables treated as constants]

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given differential eqn be an exact

The given diff. eqn. can be written as

$$axdx + \underline{hydx} + gdx + \underline{hxdy} + bydy + f dy = 0$$

$$\Rightarrow axdx + gdx + bydy + f dy + h(dx + dy) = 0$$

Integrating, we get

$$\frac{ax^2}{2} + gx + \frac{by^2}{2} + fy + hxy = k$$

$$\Rightarrow \boxed{ax^2 + by^2 + 2gx + 2fy + 2hxy = C} \quad [2k = C]$$

where C is const. of integration.

Which is the required solution.

Q.3) Solve: $xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$

Solution:— The given equation can be written as

$$(x + \frac{a^2y}{x^2 + y^2})dx + (y - \frac{a^2x}{x^2 + y^2})dy = 0$$

$$\text{Let } M = x + \frac{a^2y}{x^2 + y^2} \text{ and } N = y - \frac{a^2x}{x^2 + y^2}$$

$$\text{Now, } \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{a^2y}{x^2 + y^2} \right) = \frac{a^2}{x^2} \left(-\frac{1}{y^2} \right) = -\frac{a^2}{x^2 y^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(-\frac{a^2}{x^2} \right)$$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)a^2 - a^2y \cdot 2y}{(x^2 + y^2)^2} = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\text{and } \frac{\partial N}{\partial x} = -\frac{a^2(x^2 + y^2) + 2a^2x^2}{(x^2 + y^2)^2} = \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given eqn. be an exact.

Integrating M w.r.t x regarding y as constant,
we get

$$\frac{1}{2}x^2 + a^2y \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) = \text{or, } \frac{1}{2}x^2 + a^2 \tan^{-1}\left(\frac{x}{y}\right)$$

In N, term free from x is y whose integral is $\frac{1}{2}y^2$
Hence solution is

$$\frac{1}{2}x^2 + a^2 \tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}y^2 = C$$

$$\text{i.e. } x^2 + 2a^2 \tan^{-1}\left(\frac{x}{y}\right) + y^2 = K$$

which is the required solution.

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